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A DISCUSSION BY SYNTHETIC METHODS OF TWO PROJECTIVE PENCILS OF CONICS

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* A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in the College of Natural Sciences of the University of California.

INTRODUCTORY OUTLINE

The Problems Treated in the Discussion

The following study of pencils of conics falls naturally into four parts, concerned each with the discussion of a particular problem.

The first part—with the exception of its introduction, which deals with the geometry of a one-to-one correspondence between pencils of conics—is concerned with the locus of intersections of corresponding elements of a pencil of rays of the first class and a pencil of conics. This locus is shown to be a cubic curve; and, from the construction, a discussion of the one-to-two involutory correspondence of points on a line is obtained. In this problem, Pascal's Theorem is found to be a useful tool; and, indeed, in the succeeding discussions, the method of attack is often by means of this theorem.

The second part is concerned with the locus of intersections of corresponding elements of two projective pencils of conics. This locus is shown to be the general quartic curve, and conditions are obtained for the double points. The solution is found to depend on the locus of intersections of two involutions of rays where there is a one-to-one correspondence between pairs of rays of the two.

This locus is studied in the third part and is shown to be a quartic curve with two double points. When the involutions are in half-perspective position, the locus is the single-branched cubic.

From the discussion of the problem of part three, a locus problem is suggested which is solvable with the aid of Pascal's Theorem, and which gives rise to a certain quadratic transformation of the plane which is studied by analytic methods. This is the essence of the fourth part.

I

The Cubic as a Locus of Intersections of a Pencil of Rays and a Pencil of Conics Projective to it

Introduction

In the following discussion, the term "pencil of conics" will be used to designate the totality of conics that can be constructed passing through four arbitrary fixed points in a plane. When two such pencils of conics are so related that there is a one-to-one correspondence between the conics of one pencil and those of the other, the two pencils will be said to be projective to each other. The geometrical construction used to determine this one-to-one correspondence will be set up as follows:

If an arbitrary line be drawn through one of the fixed points of a pencil of conics, it is obvious that every conic of the pencil will meet this line in one point besides the fixed point, which is common to all. Conversely, every point

of the line will determine a single conic of the pencil passing through it. In particular, the fixed point of the pencil of conics considered as a point of the arbitrary line will be assumed to determine the conic of the pencil which is tangent to the line at the fixed point of the pencil. It will be shown later that two such point-rows are projective in the ordinary sense.

If this line, considered as a point-row, be projectively related to a pencil of rays of the first class, there will be a one-to-one correspondence between the rays of the pencil of rays and the conics of the pencil of conics. In this case the pencil of rays and the pencil of conics are said to be projective to each other. If, in a second pencil of conics, an arbitrary line be similarly drawn through one of the fixed points, the conics of the two pencils of conics can be put in one-to-one correspondence by merely considering the two arbitrary lines as projective point-rows. This construction, as outlined, is employed throughout the following discussion.

1. *Locus of Intersections of Pencil of Rays and Pencil of Conics*

The first problem to be discussed is the following: Required, the locus of intersections of corresponding elements of a pencil of rays of the first class projectively related to a pencil of conics.

Consider a pencil of conics through the points A_1, A_2, A_3, and A_4 (see fig. 1), and a pencil of rays through S, given projectively related to it. The ray by which the projectivity is set up is denoted by A_1Q. Hence, the pencil of rays at S is projectively related to the ray A_1Q, considered as a point-row. Consider an arbitrary cutting ray l in the plane. It may be taken as a point-row perspective to the pencil of rays S, and will therefore be projective to the point-row A_1Q. But points on l where a ray of S meets its corresponding conic are points of the required locus. Now, if a ray revolves about S, its corresponding point P moves along l, cutting out a point-row projective to the point-row Q cut out on A_1Q by the corresponding conic. If P falls on the conic of the pencil determined by Q, P is a point of the locus. In other words, whenever the six points P, Q, A_1, A_2, A_3, and A_4 lie on a conic, P is a point of the locus, and the six points named will satisfy Pascal's Theorem regarding a hexagon inscribed in a conic.

Number the points as indicated in the figure. Call the point of intersection of 12 and 45, L; of 23 and 56, M; of 34 and 61, N.

As P moves along l, 34 revolves about the fixed point 3, cutting out a point-row N on the fixed ray 61, perspective to the point-row P. Similarly L cuts out on a_1 a point-row perspective to P. Likewise, since the point-row Q is projective to P, M cuts out a point-row on the fixed ray 56 perspective to Q, and hence projective to P.

Hence, the point-rows L, M, and N are projectively related to one another, and there will, consequently, be at most three rays which pass through corresponding points of the three point-rows. For these three cases, P lies on the conic of the pencil determined by Q, and is a point of the locus. There are at most three such points on an arbitrary ray l. Hence, the

Theorem: The locus of intersections of corresponding elements of a pencil of rays of the first class and a pencil of conics projectively related to it, is a point-row of the third order.

That this locus is the general plane cubic is easily demonstrated both analytically* and synthetically.†

It should be noted that the four fixed points of the pencil of conics are on the locus; since one of the intersections of the line SA_1, for example, with its corresponding conic must be at A_1. Similarly, S is on the locus, since the conic of the pencil which passes through S must meet its corresponding ray there.

By the conditions of the problem, we observe that to any point A of l (see fig. 2) considered as a point of intersection with a ray of S, there correspond two points B_1 and B_2, considered as the points of intersection with l of the conic corresponding to that ray; and that, conversely, to this same pair of points B_1 and B_2 corresponds back again the starting-point A. Hence the following

Theorem: In an involutory one-to-two correspondence of points on a line, there are at most three points where corresponding points are coincident.

II

The Quartic Q as the Locus of Intersections of Two Projective Pencils of Conics

2. *Reference to Analytical Discussion*

The next problem to be studied is that of the locus of intersections of corresponding elements of two projective pencils of conics.

This locus is easily shown analytically to be the general quartic curve.* Let us denote it by Q. That it is a point-row of the fourth order will presently be demonstrated synthetically.

3. *Point-rows through a fixed point of a Pencil of Conics*

Before proceeding to this, however, let us examine a few properties of the figure, and enunciate a theorem that is of use later on.

Theorem: If, in a pencil of conics through four fixed points, rays are passed through the several fixed points, the point-rows described by the other intersections of the conics with the rays are projective to one another.

In fig. 3, represent the fixed points of the pencil of conics by A_1, A_2, A_3, and A_4, and two of the fixed rays by a_1 and a_3. Consider any conic of the pencil, and call the points in which it intersects a_1 and a_3, 2 and 5 respectively. The six points 2, 5, A_1, A_2, A_3, and A_4 must satisfy Pascal's Theorem. Hence, numbering the points as indicated in the figure, we have 12 and 45 intersecting at L, 23 and 56 at M, and 34 and 61 at N. Both L and N are fixed points, therefore this Pascal line of the pencil of conics is fixed. As 2 moves along the ray a_1

* See Emch, *Introduction to Projective Geometry and its Application*, p. 182.

† Schröter, *Theorie der Ebenen Curven dritter Ordnung*, p. 58.

* See Emch, *loc. cit.*, p. 181.

determining the various conics of the pencil, it projects to A_4 in a pencil of rays, giving a point-row M on the Pascal line perspective to the point-row 2. The point-row M projects to A_2 in a pencil of rays giving a point-row 5 on a_3 perspective to the point-row M and, hence, projective to the point-row 2. Therefore, the pencil of conics cuts out on a_1 and a_3 point-rows that are projectively related to each other. This may be extended to include rays through the other fixed points, or several rays through the same fixed point.

4. *Double Points of Quartic*

Consider further the two pencils of conics determined by the points A_1, A_2, A_3, and A_4 and B_1, B_2, B_3, B_4 (fig. 4), projectively related to each other by means of point-rows a_1 and b_1 through A_1 and B_1 respectively.

Theorem: The eight fixed points of the two projective pencils of conics are on the locus Q of intersections of corresponding conics.

This is evident since the conic of the first set passing through B_1, for example, must meet its corresponding conic of the second set there. Similarly, for the others.

Theorem: If the two projective pencils of conics have a fixed point in common, this is a double point of the locus Q.

Call the common point (A_1B_1) (see fig. 5). The projectivity of the two pencils of conics may now be referred to two point-rows through (A_1B_1), say a_1 and b_1, which are projectively related to each other—since these are projective to any other point-rows through any of the fixed points of either pencil.

Since the projectivity between the pencils of conics is arbitrary, the point-rows a_1 and b_1 are not, in general, perspective to each other, and the point (A_1B_1) is not, in general, self-corresponding. Hence (A_1B_1) considered as a point of a_1 corresponds to another point of b_1, say R, giving (A_1B_1) as a point on the locus Q, determined by the conic of the first pencil tangent to a_1, and the conic of the second pencil through R. Considered as a point of b_1, (A_1B_1) corresponds to another point of a_1, say P, and hence gives (A_1B_1) as a point on the locus Q determined by an entirely different pair of conics, the conic of the first pencil through P, and the conic of the second pencil tangent to b_1. Hence, (A_1B_1) occurs twice on the locus or, is a double point. There may be in this way as many as three double points. When there are three, the curve is evidently unicursal, since there is but one movable point of intersections of corresponding conics, and the correspondence is continuous. If there are four common fixed points, the locus degenerates, in general, either into these fixed points, or into two conics through them.

5. *Synthetic Discussion of Order of Q*

With this introduction, we proceed to the synthetic discussion of the order of the locus.

In figure 6, denote by $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ the fixed points of the two pencils of conics, by a_2 and b_2 the point-rows determining the projectivity of the two pencils of conics, and by l an arbitrary cutting ray of the plane. Let a

pair of corresponding conics, determined by P of a_2 and R of b_2, cut l in A' and A'', and B' and B'' respectively. As P moves along a_2, A' and A'' will describe an involution of points on l; R will move along b_2, and B' and B'' will describe another involution of points on l. There is a one-to-one correspondence between pairs of points on l set up in this way, such that to any pair of points on l determined by a conic of one pencil corresponds a pair of points determined by the corresponding conic of the other pencil. Since the pencils of conics are projectively related, the correspondence of point-pairs on l is involutory.

Hence, our problem is reduced to that of discovering how many coincidences of corresponding points of an involutory two-to-two correspondence of points on a line can occur. For, if a point of l be at once a point of both involutions, it is an intersection point of corresponding conics and, therefore, a point of the locus. Join the points A' and A'' to an arbitrary point, say A_1, and B' and B'' to B_1. Now, as P moves along a_2 and R moves along b_2, projective to P, we shall have an involution of rays at A_1 and also one at B_1, with a one-to-one correspondence between pairs of rays of the two involutions. The locus of intersections of corresponding pairs of rays of these two involutions—call it L—will meet l in points where corresponding conics intersect; in other words, in points of the original locus Q of intersections of corresponding conics. Hence, the order of L is the same as that of the original locus, and we shall confine our attention for the moment to L.

6. *Proof of Order of L*

Theorem: The locus of intersections of corresponding pairs of rays of two involutions of rays, where there is a one-to-one correspondence between the pairs of rays of the two involutions, is a point-row of the fourth order with two double-points; or, a quartic curve of deficiency one.

In figure 7, denote by I_1 and I_2 the centers of the involutions of rays and let λ be an arbitrary cutting ray of the plane. Construct through I_1 and I_2 any conic—call it Γ—which is tangent to λ. There will be a double infinity of such conics. Now, the rays of I_1, for example, will cut out an involution of points on λ. Draw from each of these points the remaining tangent to Γ. Now, to a given pair of rays of I_1, there will be a pair of points on λ, say G_1 and G_2, and hence a pair of tangents to Γ, say g_1 and g_2. If four points G_1 of λ be taken, the four points of tangency of the four tangents g_1 will project to any point of Γ in four rays with the same anharmonic ratio as the points G_1, since the points G_1 are chosen on a tangent to Γ. Hence, the points of tangency of the various pairs of tangents of g_1 and g_2 will constitute an involution of points on Γ. Now, the rays joining corresponding points of an involution of points on a conic pass through a point,* say S_1. Hence, the pencil of rays S_1 will determine our involution of points on Γ, and, consequently, our involution of points on λ, determined by the rays of I_1. Of course, S_1 varies with the different conics Γ that may be taken. Similarly, there will be a pencil of rays S_2, determined by

* See Reye, *Geometrie der Lage,* p. 147.

tangents drawn to Γ from the involution of points H_1 and H_2, on λ, determined by the involution of rays I_2. Since there is a one-to-one correspondence between the line-pairs of the two involutions, there will be a one-to-one correspondence between the rays of S_1 and S_2. The point of intersection of corresponding rays of S_1 and S_2 will therefore describe a conic, say Σ, which will intersect Γ in, at most, four points. These points are coincident points of the two involutions of points on Γ and, consequently, since tangents g and h coincide here, the tangents to Γ at these points will meet λ in points where G and H coincide; that is, in points of L. There can be at most four such points on an arbitrary λ. Hence, the first part of the theorem above is established, and the two following theorems may be added as direct consequences of the discussion,—

Theorem: In a two-to-two involutory correspondence of points on a line there are at most four points where corresponding points coincide.

Theorem: The locus of intersections of corresponding conics of two projective pencils of conics is a point-row of the fourth order.

The conditions for double points have been established above. Hence, the locus is the general quartic curve.

III

Discussion of the Quartic L with two Double Points

7. *Deficiency of L*

In article 6 of the preceding part, we have shown that L is a point-row of the fourth order. Let us proceed to a discussion of this quartic beginning with the determination of its deficiency, which may be established as follows. Consider the ray I_1I_2 as a ray of the involution I_1. Considered as a ray I_1G_1, it meets its corresponding ray I_2H_1 at I_2; likewise, considered as a ray I_1G_2, it meets its corresponding ray I_2H_2 at I_2. I_2H_1 and I_2H_2 are in general different rays. Hence I_2 occurs twice on L or, is a double point.

Similarly with I_1. A third double point does not in general exist; but conditions for its existence are easily determined.

A pair of rays of I_1 has its corresponding pair of rays of I_2. The four intersection points—say P, Q, R, S—are points of L. If the two rays of I_2 should fall together as a double ray of the involution at I_2, the points P and Q, and R and S would fall together in pairs as indicated, and the rays of I_1 corresponding to a double ray of I_2 would be tangents to L. There are not more than two double rays of I_2. Hence,

Theorem: From either double point of L at most four tangents may be drawn to L. These are the rays of the involution at one double point corresponding to the double rays of the involution at the other double point.

If a double ray of I_1 corresponds to a double ray of I_2, they are each tangent to L at their point of intersection, and their point of intersection is, consequently, a double point of L. Hence,

Theorem: If a double ray of I_1 corresponds to a double ray of I_2, L has three double points, viz: I_1, I_2, and the point of intersection of the corresponding double rays.

8. *Bearing of Quadratic Transformation Theory*

Before discussing other cases of L, let us consider the bearing of the quadratic transformation theory of the plane on this discussion. Ordinarily, if two involutions of rays, I_1 and I_2, be given, and the point of intersection P of a ray of I_1 and one of I_2 move on a point-row of order n, the intersection point R of the corresponding rays of I_1 and I_2 will move on a point-row of order $2n$. I_1 and I_2 will be vertices of a fundamental triangle of the transformation, of which the third vertex, say I_3, will be the center of an involution of rays that may be used in place of either I_1 or I_2 in the construction. Now, each time the point P of the point-row of order n passes through one of the points I_1, I_2, or I_3, the point R of the point-row of order $2n$ describes a straight line as part of its locus and the order of the remainder is reduced by one. In the locus L, under discussion, the points P and R traverse the same point-row, a point-row of order four with two double points. Comparing this with the theory mentioned above, we find it to be consistent; for, if P move along L, Q will move on a point-row of order eight. However, P passes through I_1 twice and I_2 twice. Hence, the locus of Q degenerates into a point-row of the fourth order and doubly into the lines in the quadratic transformation which correspond to the points I_1 and I_2 of the fundamental triangle.

9. *Degeneracy of L with Double Rays Corresponding*

Certain degenerate cases of L are interesting and present themselves naturally. Suppose, first, that each double ray of I_1 corresponds to a double ray of I_2. Let the first double ray of I_1 meet its corresponding double ray of I_2 at A, and the second double ray of I_1 its corresponding double ray at B (see fig. 8).

By previous reasoning I_1, I_2, A and B are double points of L, hence—with four double points—degeneracy is to be expected. The ray AB has the same involution of points described on it by the rays of I_1 and I_2, since the double points A and B of the involutions are the same. Hence, the corresponding rays of I_1 and I_2 will meet on AB and it is a part of the locus. Let the points of intersections of corresponding pairs of rays of I_1 and I_2 be P, Q, R, and S. Two of these, say P and R, are obviously on AB. Now, as P moves along AB, it is always an intersection point of corresponding rays of I_1 and I_2. As it crosses I_1I_2, the point of intersection of the corresponding rays of I_1 and I_2 is indeterminate, and I_1I_2 is thus a part of the locus. The remainder is a conic through I_1, I_2, A, and B, and there are not only four but five double points to the locus in this case.

10. *Single-branched Cubic*

Another and more important case of degeneracy is that in which the ray I_1I_2 is self-corresponding, but is not a double ray of either involution. In this case, the locus L degenerates into the ray I_1I_2 and a point-row of the third order. This last is a single-branched cubic and has been discussed by Schröter from this point of view. This position of two involutions of rays is termed by him "half-perspective position."*

In figure 9, a case of this type is shown. From T_1, a point on a single-branched cubic, tangents t_1 and t_2 are drawn to the curve. (There are always points T_1 from which this is possible.) Calling the points of tangency A_1 and A_2, we know that the ray A_1A_2 meets the cubic in another point, say T_2, which is conjugate to T_1. If, now, any point P of the cubic be joined to A_1 and A_2, the lines so drawn will each meet the cubic in one other point. Call these B_1 and B_2, and join them to T_1 and T_2. As P moves along the curve, the rays T_1B_1 and T_1B_2 will give an involution of rays at T_1, and T_2B_1 and T_2B_2, an involution of rays at T_2. Since T_1T_2 is self-corresponding, it is part of the locus. The ray A_1A_2 is a double ray of T_2; hence, the corresponding rays of T_1 should be tangents to the curve as, indeed, they are.

11. *Conditions of Tangency of a Line to L*

Let us turn again to a discussion of the properties of L as revealed in figure 7. If the conic Σ, described by the intersection point of corresponding rays of S_1 and S_2 is tangent to the conic Γ, two of the points of intersection of the conics will be coincident, and two of the points of intersection of λ with L will be coincident. Hence λ will either be a tangent to L or a secant through a double point. That this last state of affairs might occur is readily shown.

In figure 10, let the double rays of I_1 meet λ in A and B, and the double rays rays of I_2 in C and D. Then, since the tangents to Γ from these points determine the double points of the two involutions of points on Γ, it is obvious, for instance, that the tangent to Γ from A joining, as it does, two corresponding points on Γ, contains S_1. Similarly, the tangent from B contains S_1. S_1 is therefore determined as the intersection point of the tangents to Γ from the points where the double rays of I_1 intersect λ. S_2 may be determined in like manner. Now, if one double ray of I_1 correspond to a double ray of I_2, their point of intersection A is a double point of L. Consider any cutting ray λ through this point, and construct the conic Γ as before. A tangent to Γ from this double point contains both S_1 and S_2. Now, since the point A is a self-corresponding double point of the involutions of points on λ, the common ray S_1S_2 of the pencils of rays S_1 and S_2 is self-corresponding, and the conic Σ degenerates into two lines, one of which is S_1S_2, the tangent to Γ from A. This is obviously true for any ray λ through A and, hence, Σ and Γ are tangent if λ be tangent to L, or pass through one of its double points.

* See Schröter, *Theorie der Ebenen Curven dritter Ordnung*, p. 148.

The same remarks apply to the general quartic determined by two projective pencils of conics, since the intersections of λ with the general quartic are the same as its intersections with the L determined for that particular cutting line. If the cutting line should pass through one of the double points of L, a single involution only of points would be determined upon it, and our reasoning regarding the coincidences of intersection points of Q and L with λ breaks down. Since the selection of double points for L is arbitrary, this difficulty is obviated by moving them.

As a consequence of the identity of the intersections of L with λ and of the general quartic with λ, we may revolve λ about a fixed point, and enunciate the following

Theorem: The general quartic curve may be described as the locus of intersections of a pencil of rays and of a pencil of quartics of deficiency one with arbitrary, fixed double points.

12. *One-to-Two Involutory Correspondence of Points on a Line from this Viewpoint*

Before leaving the discussion of L, let us apply this machinery to the discussion of the one-to-two involutory correspondence of points on a line, previously discussed with the aid of Pascal's Theorem.

In figure 11, let I_1 be the center of a pencil of rays of the first class, and I_2 the center of an involution of rays. Select any cutting ray λ and construct a conic Γ, containing I_1 and I_2, and tangent to λ. The points of tangency of tangents to Γ from the points H_1 and H_2, where pairs of rays of I_2 cut λ, will give an involution of points on Γ. The rays joining corresponding points of this involution will pass through a point, say S_2. The points of tangency of the tangents from the points G of λ where rays of I_1 meet λ will project to I_1 in a new pencil of rays projective to the original one. By the conditions of the problem, the new pencil of rays at I_1 and the pencil of rays at S_2 are projective to each other. They determine a conic Σ which cuts Γ in at most four points, one of which is I_1. Tangents to Γ at the other three points of intersection of Σ and Γ meet λ in points where corresponding points of the one-to-two involutory correspondence are coincident. The tangent at I_1 does not in general cut λ in such a point, for the tangent (from a point of λ) whose point of tangency is I_1, does not uniquely determine a ray of the new pencil at I_1, since any ray through I_1 answers the requirements. Hence the following

Theorem: The locus of intersections of corresponding elements of a pencil of rays of the first class and an involution of rays, where there is a one-to-one correspondence between the rays of the first and the pairs of rays of the second, is a point-row of the third order with one double-point; or, a unicursal cubic curve.

I_2 is the double point, since the ray I_1I_2, considered as a ray of I_1, meets two rays of I_2 at I_2. Hence, I_2 occurs twice on the locus, or, is a double point. I_1 is obviously also on the locus, since the ray I_2I_1 of I_2 meets its corresponding ray of I_1 at I_1. The last theorem may be stated inversely in the form in which it is given when approached from the other side.

*Theorem: The points of intersection with the curve of rays through any point of a unicursal cubic project to the double point in an involution of rays, the double rays of which are determined by the tangents from the arbitrary point to the curve. The rays of the involution corresponding to the ray joining the arbitrary point to the double point are the tangents of the curve at the double point.**

13. *Four-to-Four Transformation of the Plane*

The constructions of figures 7 and 10 by which L was discussed, contain and suggest several problems and loci. The points S_1 and S_2 are determined by constructing a conic through two points and tangent to a given line. Let us inquire whether it is possible to choose S_1 arbitrarily and, if so, how many points S_2 will there be corresponding to a given S_1. We note that with a given S_1, Γ is determined as a conic which shall be tangent to S_1A, S_1B, and λ, and shall, in addition, contain the points I_1 and I_2. There are, in general, four such conics.† For a given conic, it is obvious that S_2 is uniquely determined. Hence, to a given S_1, arbitrarily chosen, there are four S_2's, and conversely. This gives a four-to-four transformation of the plane, which may be called "one-quarter involutory," since any one of the points S_2 determined by an arbitrary S_1, gives back this same S_1 and three others.

14. *Two-to-Two Semi-involutory Correspondence of Two Pencils of Rays*

Suppose the point of tangency of Γ to λ be fixed and be denoted by K. Then the figure furnishes an example of what may be termed a "semi-involutory" two-to-two correspondence of rays, as follows,—suppose an arbitrary ray of A be chosen. On this ray there are in general two points S_1, since two conics of the pencil are in general tangent to the ray of A chosen. The conics Γ now constitute a pencil, since they all pass through I_1 and I_2 and are tangent to λ at K. Also, each conic determines a ray of B, tangent to it, and hence an S_1. To either of the rays of B so determined, two conics of the pencil are tangent, one of which is the conic tangent to the original ray of A. Hence, to each ray of A, there are two rays of B; and, to each of these rays of B, there are two rays of A, one of which is the ray of A with which we started.

An easy geometrical example of this is obtained by using either two points and two rays which do not contain them, or two points and a non-degenerate conic which does not contain them (see figures 12 and 13). To the ray a_1 of A, there are two rays b_1 and b_2 of B, and to the ray b_1 of B there are two rays a_1 and a_2 of A—one of which is a ray of A which determined the ray b_1 of B.

In figure 12, the rays joining A and B to the point of intersection R of the rays p and q of the construction are corresponding double rays. In figure 13, the tangents from A to the conic have double rays of B corresponding to them,

* Proved by D. N. Lehmer, "Constructive Theory of Unicursal Cubic by Synthetic Methods," *Transactions of American Mathematical Society*, 1902.

† See Salmon, *Conic Sections* (ed. 10), p. 389.

and, similarly, the tangents from B to the conic have double rays of A corresponding to them. This correspondence can be studied further from this point of view.

IV

Locus Problem Suggested by the Discussion of the Quartic L.

15. *Locus Problem Synthetically*

By altering slightly the construction and interpretation of S_1, we obtain a problem whose solution is interesting as furnishing another example of the method of using Pascal's Theorem in problems involving pencils of conics. The power of this method is obvious from the several uses of it already made in this discussion. Suppose that S_1 is a point determined as the point of intersection of tangents to Γ at the points where it meets the double rays of I_1. If the point of tangency K of Γ be fixed and Γ be determined as a conic through I_1 and I_2 and tangent to λ at K, what is the locus of S_1 as the various conics Γ of the pencil are taken? This problem is not directly connected with the study of L previously undertaken, but comes in naturally as a problem connected with this particular pencil of conics.

In figure 14, let Γ cut AI_1 in R, and BI_1 in R'. Call the tangents at R and R', α and β respectively. Their points of intersection is S_1. The points I_1, I_2, K and R must satisfy Pascal's Theorem. Number them, and call the intersection of 12 and 45, L; of 23 and 56, M; of 34 and 61, N, as indicated in the figure. The points L, M, N are on the Pascal line, and N is a fixed point of this line for the given pencil of conics. Similarly, construct the Pascal line $L'M'N'$, replacing R in the construction with R'. As R moves along AI_1, it cuts out a point-row determining the conics of the pencil. This point-row projects to I_2 in a pencil of rays which determines on KI_1 (a fixed line) a point-row L perspective to the point-row R. The point-row L of KI_1 projects to N in a pencil of rays which describes a point-row M on AB, perspective to the point-row L of KI_1, and hence projective to the point-row R on AI_1. Therefore, the ray RM envelopes a conic, as R moves on AI_1, determining the various conics of the pencil. Similarly, $R'M'$ envelopes another conic. Since there is a one-to-one correspondence between the tangents RM and $R'M'$, S_1 is determined as the locus of intersections of two projective pencils of rays of the second class. This curve is the unicursal quartic, and has been discussed from this point of view.* If a common ray of the two pencils is self-corresponding, it is, obviously, part of the locus, so that, if all four common rays be self-corresponding, they constitute the locus. Indeed, if three are self-corresponding, the locus consists of them and of one additional line. We shall show in this problem that the locus is degenerate and consists of the lines AB, KI_2—counted twice—and another line.

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* See Annie Dale Biddle, "Constructive Theory of the Unicursal Plane Quartic by Synthetic Methods," *Univ. Calif. Publ. Math.*, vol. 1, no. 2, 1912.

Denote by Σ the conic enveloped by RM, or α, and by Σ', the conic enveloped by $R'M'$, or β. AB and AI_1 are tangents to Σ, and AB and BI_1 are tangents to Σ'. Hence, AB is a common tangent of Σ and Σ'. Now, in the conic Σ, the points corresponding to A, considered first as a point of AB and then as a point of AI_1, are the points of tangency to Σ of AI_1 and AB, respectively. If R moves to A, RI_2 becomes AI_2 and cuts KI_1 in a point denoted by Q. NQ determines M_1 as the corresponding position of M. This construction will be referred to later. Hence, M_1 is the point of tangency of AB to Σ. Moreover, if R moves to A, the conic Γ degenerates into AB and I_1I_2, and R' moves to B. Hence, the common tangent AB of Σ and Σ' is self-corresponding and is therefore part of the locus. When R' moves to B, $R'I_2$ becomes BI_2, and intersects KI_1 in Q'. $N'Q'$ determines M_1' as the point of tangency of AB to Σ'.

If R moves to N, L moves to K, M moves to K, and Γ degenerates into KI_1 and KI_2. Since R is at N and M at K, KN is a tangent of Σ, being a ray RM. Since, when R is at N, Γ degenerates into KI_1 and KI_2, R' is either at I_1 or N', as it is always on the conic Γ. If R' is at I_1, M' is at B, and $R'M'$ is not a tangent to Γ. As this is contrary to hypothesis, R' is at N'. Consequently, L' is at K, and M' is at K. Hence KN is a tangent of Σ', that is, KN is a self-corresponding common tangent of Σ and Σ', and is therefore a part of the locus of S_1. We shall apply Brianchon's Theorem in the following way to determine its points of tangency to Σ and Σ'.

Let AB, BC, and CA of figure 15 be three tangents to a conic and let S and T be the points of tangency of CA and AB, respectively. Then R, the point of tangency of BC, is found by drawing through A, a ray passing through the point of intersection of BS and CT.

In figure 14, the point of tangency of AB to Σ has been found to be M_1. The point of tangency of AI_1 is found by moving M to A. If this is done, L moves to I_1 and R moves to I_1. Hence AI_1 is tangent to Σ at I_1. To discover the point of tangency of KN to Σ, apply Brianchon's Theorem as follows, remembering that R is at N. Join M_1 to N and K to I_1, these lines meeting in Q. AQ cuts KN in the required point of tangency, which is seen to be I_2, from the way in which Q was previously determined.

To discover the point of tangency of KN to Σ', it is necessary to find the point of BI_1 to Σ'. If M' moves to B, L' moves to I_1, and R' moves to I_1. Hence, BI_1 is tangent to Σ' at I_1. The point of tangency of AB to Σ' has already been discovered to be M_1'.

Now, to discover the point of tangency of KN to Σ', apply Brianchon's Theorem, remembering that R' is at N'. Join M_1' to N' and K to I_1, giving Q'. $Q'B$ cuts KN in the required point, which is seen to be I_2. Hence, Σ and Σ' are tangent to each other at I_2, and KI_2 is a double common tangent which is self-corresponding.

Hence, finally, the locus of S_1 is composed of the lines AB, KI_2 (counted twice) and another. That is to say, S_1 moves in general on a straight line.

Similarly, if an S_2 be determined by tangents to Γ at the points where the double rays of I_2 cut Γ, S_2 will move on a line. As different points K of λ are taken as points of tangency for Γ, S_1 and S_2 will move on lines which will correspond in pairs. Of this, more will be said later.

16. *Locus Problem Analytically*

The analytic discussion of this problem reveals a further property of these rays. This is inserted temporarily, as a complete synthetic discussion has not yet been obtained.

In figure 16, denote by I_1 and I_2 the fixed points, by AI_1 and BI_1 the fixed lines through I_1 and by K the fixed point of tangency of conics through I_1 and I_2 to an arbitrary line 1. The tangents α and β at M and N will determine S_1. Choose I_2, I_1, and K as the points $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$ respectively. The equation of Γ may be put in the form

$$yz + zx + \lambda xy = 0$$

AB is the tangent at $(0:0:1)$. Hence, its equation is

$$AB \qquad x + y = 0.$$

Choose for coördinates of A and B, $(1:-1:\mu)$ and $(1:-1:\nu)$ respectively.

The equations of AI_1 and BI_1 are:

$$\begin{aligned} AI_1 &\qquad z - \mu x = 0 \\ BI_1 &\qquad z - \nu x = 0. \end{aligned}$$

Hence, the coördinates of M are $(\mu + \lambda) : -\mu : \mu(\mu + \lambda)$. The coördinates of N, similarly, are $(\nu + \lambda) : -\nu : \nu(\nu + \lambda)$.

This gives as the equation of α,—

$$[\mu(\mu + \lambda) - \mu\lambda]x + (\mu + \lambda)^2 y + \lambda z = 0$$

or

$$\mu^2 x + (\mu + \lambda)^2 y + \lambda z = 0.$$

Similarly, the equation of β is

$$\nu^2 x + (\nu + \lambda)^2 y + \lambda z = 0.$$

The coördinates of S_1, the intersection point of α and β, reduce to—

$$\mu + \nu + 2\lambda : -(\mu + \lambda) : \lambda\mu + \lambda\nu + 2\mu\nu.$$

Let the absolute coördinates of S_1 be (x_1, y_1, z_1); then

$$\begin{aligned} \zeta x_1 &= \mu + \nu + 2\lambda && (1) \\ \zeta y_1 &= -(\mu + \lambda) && (2) \\ \zeta z_1 &= \lambda(\mu + \nu) + 2\mu\nu && (3) \end{aligned}$$

whence

$$(\mu + \nu)^2 x_1 + (\mu - \nu)^2 y_1 - 2(\mu + \nu) z_1 = 0$$

results as the equation of the locus of S_1. This represents a line which cuts

$$x + y = 0$$

in the point $\left(1:-1:\dfrac{2\mu\nu}{\mu + \nu}\right)$.

Now the line I_1I_2 cuts

$$x+y=0$$

in the point $(1:-1:0)$.

The harmonic conjugate of $(1:-1:0)$ with respect to

$$A(1:-1:\mu)$$

and

$$B(1:-1:\nu)$$

is

$$(1:-1:\frac{2\mu\nu}{\mu+\nu}).$$

Hence the following

Theorem: As the point of tangency K of Γ *moves along AB, the line which is the locus of* S_1 *revolves about the harmonic conjugate, with respect to A and B, of the intersection of* I_1I_2 *and AB. If a similar construction for* S_2 *be made, the fixed points on the tangent AB being C and D, the line which is the locus of* S_2 *will revolve about the harmonic conjugate, with respect to C and D, of the intersection of* I_1I_2 *and AB.*

17. *Quadratic Transformation of Plane Resulting therefrom*

If, now, the point of tangency of Γ to AB be allowed to move and S_1 be chosen at random, we inquire as to the number of points S_2 to a given S_1, and, also, regarding the locus of S_2 when S_1 moves, for instance, on an arbitrary line.

In figure 17, let I_1, I_2, and A be the vertices of the fundamental triangle; the points $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$ respectively. Let the fixed rays of I_1 meet, in A and B, the fixed line AB which is tangent to Γ; and the fixed rays of I_2 meet this line in C and D. Call the tangents at the several points of intersection, α, β, γ, δ, as indicated in the figure: α and β determine S_1 and γ and δ determine S_2.

The equation of any conic through I_1 and I_2 may be written

$$az^2+byz+czx+dxy=0.$$

The equation of the fixed line 1 through A is

$$y-\lambda x=0$$

where λ is a constant.

That this line may be tangent to the conic Γ is represented by

$$(b\lambda+c)^2=4ad\lambda \qquad (1),$$

since the equation obtained by solving

$$az^2+byz+czx+dxy=0$$

and

$$y-\lambda x=0 \quad \text{simultaneously,}$$

viz:

$$az^2+b\lambda zx+czx+d\lambda x^2=0$$

must be a perfect square.

The equation of a tangent to Γ at a point $(x':y':z')$ on the curve is

$$(dy'+cz')x+(bz'+dx')y+(cx'+by'+2az')z=0.$$

The ray AI_1, whose equation is $y=0$, meets Γ at $(a:o:-c)$. Hence, the equation of α, the tangent at this point, is

$$c^2x+(bc-ad)y+acz=0.$$

Denote B by $(1:\lambda:\mu)$, since it is an arbitrary point on 1.

The line BI_1 has for its equation

$$\mu y-\lambda z=0.$$

Its point of intersection with Γ (besides I_1) is

$$[-\mu(a\mu+b\lambda):\lambda(c\mu+d\lambda):\mu(c\mu+d\lambda)]$$

Hence, the equation of β, the tangent to Γ at this point, is

$$(c\mu+d\lambda)^2x+(bc-ad)\mu^2y+(ac\mu^2+bd\lambda^2+2ad\lambda\mu)z=0.$$

The intersection point of α and β is S_1. Denote its coördinates by $(x_1:y_1:z_1)$. By absorbing whatever multiplier there may be (say ζ) into the constants of Γ, we may write

$$x_1=-(b\lambda+2a\mu) \qquad (2)$$
$$y_1=c\lambda \qquad (3)$$
$$z_1=2c\mu+d\lambda \qquad (4)$$

and the first condition:

$$(b\lambda+c)^2=4ad\lambda \qquad (1)$$

The equations (1), (2), (3), and (4) determine the parameters a, b, c, d of the conic Γ. Since only one of these is quadratic in a, b, c, and d, the others being linear, there are two conics to a given S_1, and hence two points S_2 to a given point S_1, as there is but one S_2 for a given conic.

Conversely, to a given S_2, there are two points S_1. This construction gives, then, a semi-involutory two-to-two transformation of the plane.

Suppose the point S_1 to move on an arbitrary line of the plane, let us discover the nature of the locus of S_2. Let the points where the fixed rays of I_2 meet the line 1 be $C(1:\lambda:\phi)$ and $D(1:\lambda:\rho)$, where ϕ and ρ are constants.

The equations of CI_2 and DI_2 follow:

$$CI_2 \qquad z-\phi x=0$$
$$DI_2 \qquad z-\rho x=0.$$

CI_2 meets Γ in the two points

$$\begin{cases} 0:1:0 \\ b\phi+d:-\phi(a\phi+c):\phi(b\phi+d) \end{cases}$$

DI_2 meets Γ in the two points

$$\begin{cases} 0:1:0 \\ b\rho+d:-\rho(a\rho+c):\rho(b\rho+d) \end{cases}$$

The equation of γ is

$$\phi^2(bc-ad)x+(b\phi+d)^2y+(ab\phi^2+2ad\phi+cd)z=0.$$

The equation of δ is

$$\rho^2(bc-ad)x+(b\rho+d)^2y+(ab\rho^2+2ad\rho+cd)z=0.$$

Denote S_2 by $(x_2:y_2:z_2)$, whence, by reduction

$$x_2:y_2:z_2=2d+b(\rho+\phi):2a\rho\phi+c(\rho+\phi):2b\rho\phi+b(\rho+\phi)$$

Let S_1 move on the line

$$Lx_1+My_1+Nz_1=0.$$

Its coördinates will satisfy this equation, and the several equations determining the locus of S_2 are

$$-L(b\lambda+2a\mu)+Mc\lambda+N(2c\mu+d)=0 \qquad (1)$$
$$(b\lambda+c)^2=4ad\lambda \qquad (2)$$
$$\tau x_2=2d+b(\rho+\phi) \qquad (3)$$
$$\tau y_2=2a\rho\phi+c(\rho+\phi) \qquad (4)$$
$$\tau z_2=2b\rho\phi+d(\rho+\phi) \qquad (5)$$

The parameters to be eliminated are a, b, c, d, and τ. As all the equations but one are linear in these, and the exceptional one is quadratic, the locus of S_2 is a conic. Hence, we may write the

Theorem: The locus of S_2, as S_1 moves on a line, is a conic; or, in general, the locus of S_2, *as S_1 moves on a point-row of order n, is a point-row of order $2n$.*

From equation (2), we note that $d=0$ gives the equation of a tangent to the S_2 conic. If $d=0$, we have, by elimination from (3) and (5)

$$\frac{X}{Z}=\frac{\rho+\phi}{2\rho\phi} \text{ or } 2\rho\phi x-(\rho+\phi)z=0.$$

This particular line is obtained regardless of which line S_1 moves on and is therefore tangent to all the S_2 conics. It is the line through I_2, which passes through the harmonic conjugate, with respect to C and D, of the intersection of I_1I_2 and AB.

Similarly, there is a line through I_1, which is tangent to all the S_1 conics which correspond to lines described by S_2. Hence, the following

Theorem: By means of the conics tangent to an arbitrary line and passing through two arbitrary points, through each of which two arbitrary rays are chosen, a quadratic transformation of the plane may be established. To every point S_1, there are two points S_2, and, conversely, the correspondence being semi-involutory. If S_1 move on a point-row of order n, S_2 moves on a point-row of order $2n$. All the S_2 loci are tangent to a certain invariant ray of the second of the two fixed points. Similarly, all the S_1 loci corresponding to arbitrary paths of S_2 are tangent to a certain invariant ray of the first of the two fixed points. In particular, however, if S_1 describe a ray passing through a certain point of the fixed tangent line, S_2 describes a ray passing through another certain point of the tangent line. Thus, in this quadratic transformation, there is a particular pencil of rays which goes into a pencil of rays.

VITA

I, Baldwin Munger Woods, was born in Lampasas, Texas, on September 22, 1887. I studied in the public schools of Fort Worth, Texas, until 1904, when I entered the University of Texas, from which I received the degree of Electrical Engineer in 1908. During the year 1907–08 I held the position of Assistant in Applied Mathematics in that institution.

In 1909, I was appointed John W. Mackay, Junior, Fellow in Electrical Engineering at the University of California. I held this position until January, 1910, when I was appointed Assistant in Mathematics. From July, 1910, to the present I have been Instructor in Mathematics in the University of California. In 1909, I received the degree of M.S. in Electrical Engineering from the University of California.

In the University of Texas, I studied under Professors Porter and Benedict and Mr. Rice in mathematics, and Professors Scott and Taylor in engineering. In the University of California, I have studied under Professors Stringham, Haskell, Lehmer, and Putnam in mathematics; under Professors Cory and LeConte in engineering; and under Professor Raymond in physics. To all these I wish to express my thanks,—especially to Professors Lehmer and Haskell, who, in their supervision of the present work, have been a constant source of inspiration.

On April 29, 1912, I passed the public final examination for the degree of Ph.D.

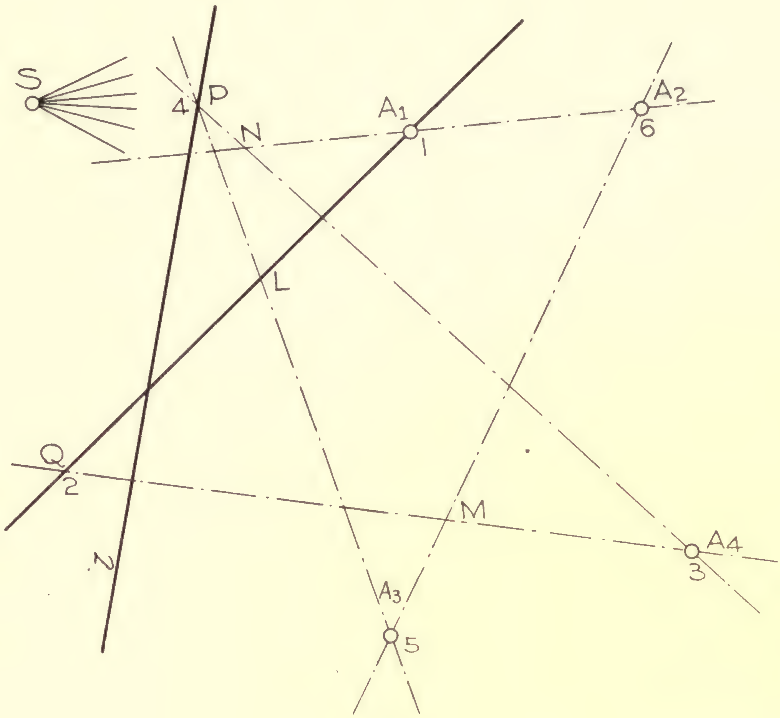


Fig. I

Fig. II

Figs. 1 and 2

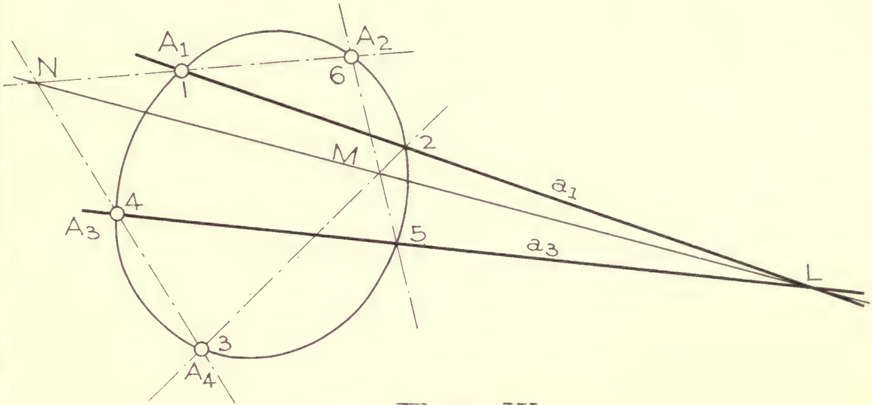


Fig. III

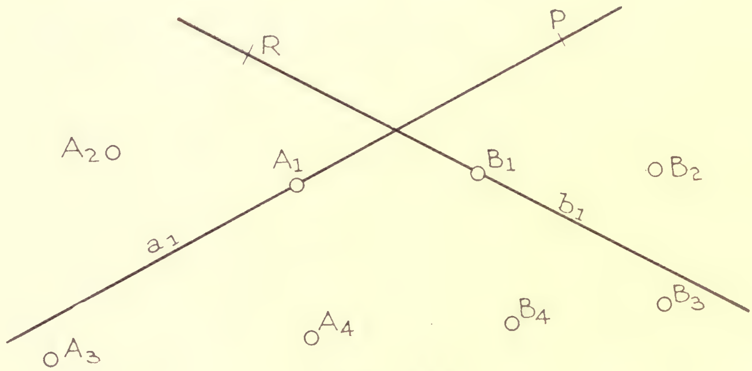


Fig. IV

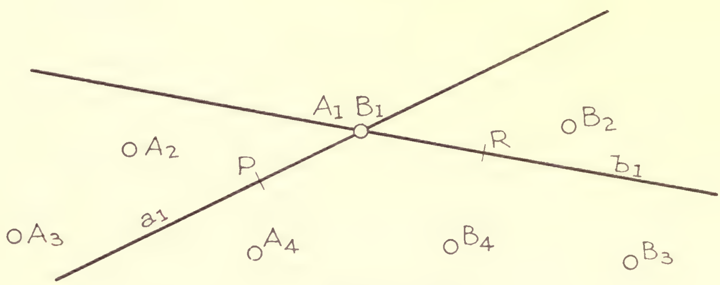


Fig. V

Figs. 3–5

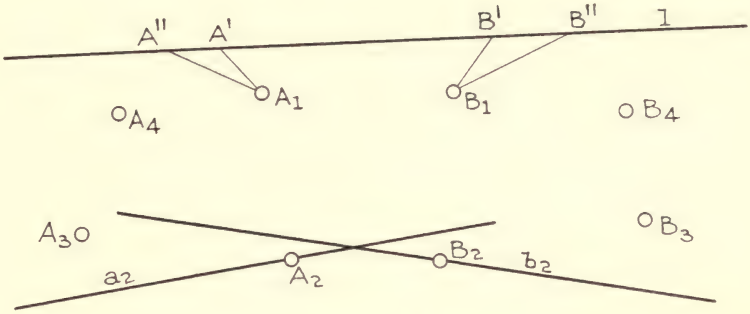


Fig. VI

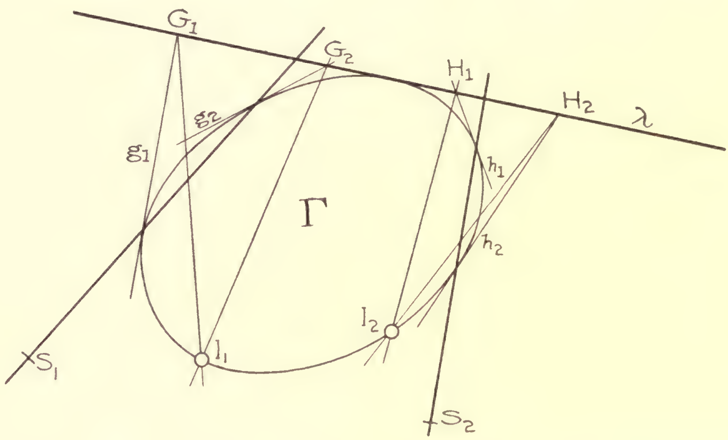


Fig. VII

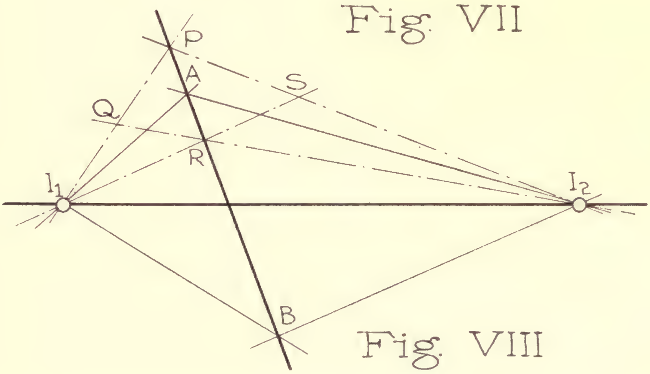


Fig. VIII

Figs. 6–8

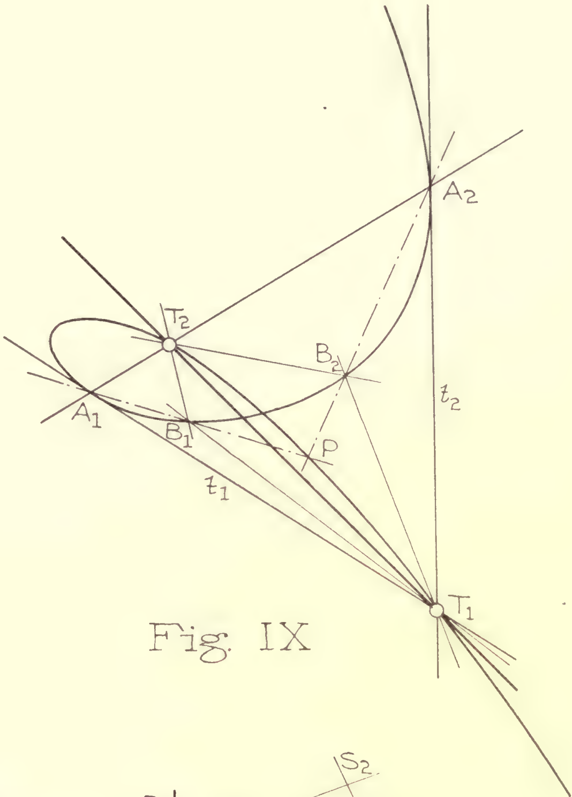


Fig. IX

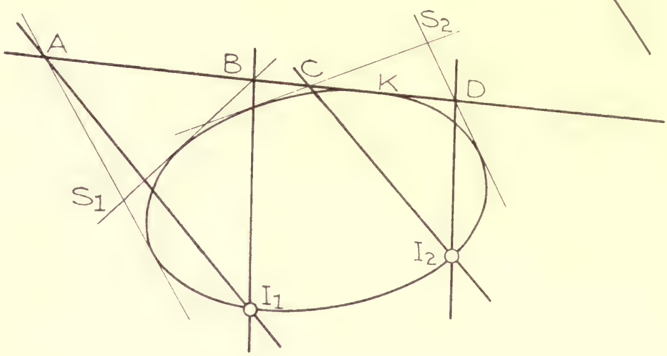


Fig. X

Figs. 9 and 10

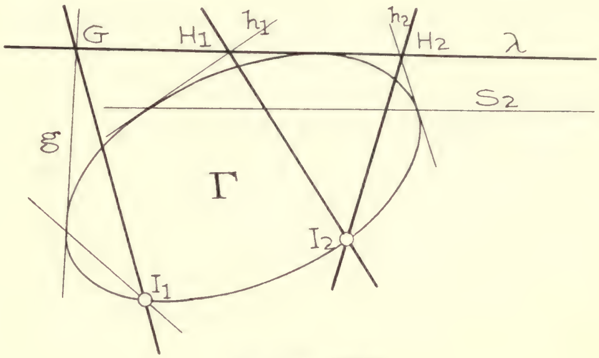


Fig. XI

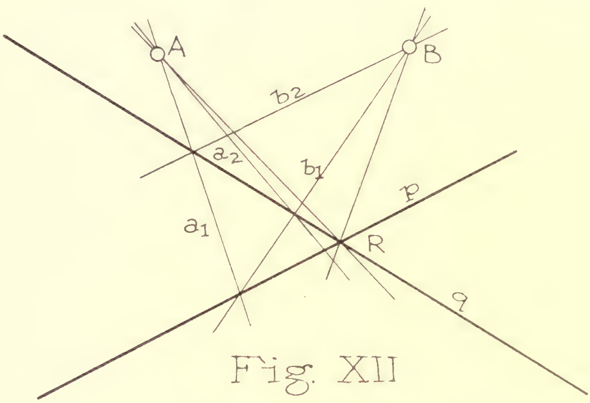


Fig. XII

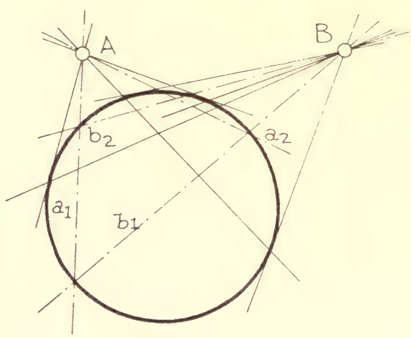


Fig. XIII

Figs. 11–13

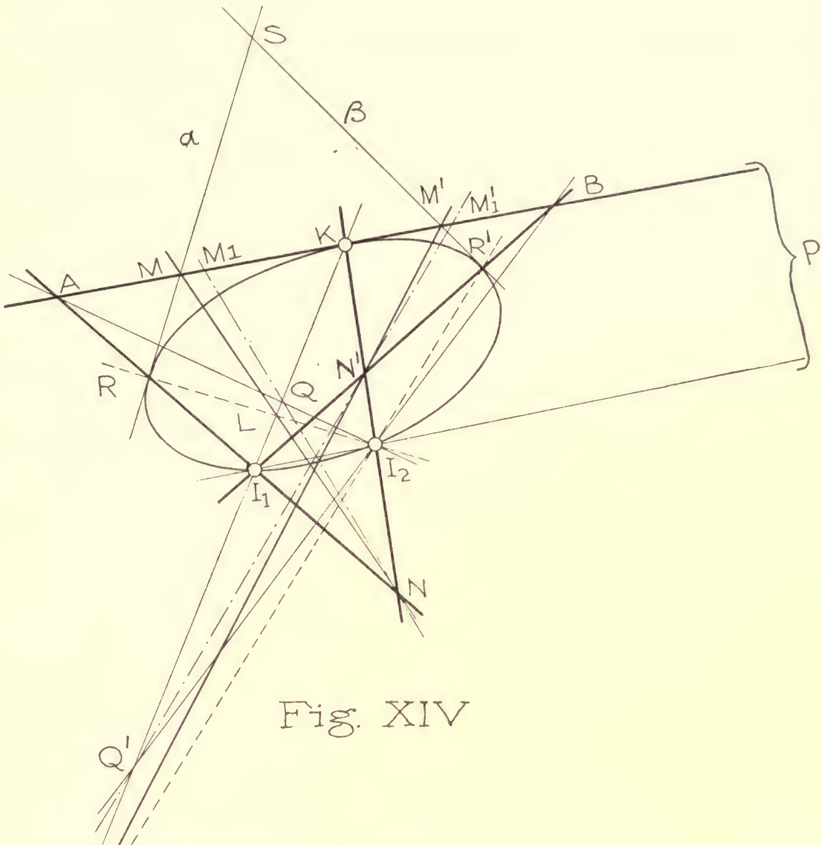


Fig. XIV

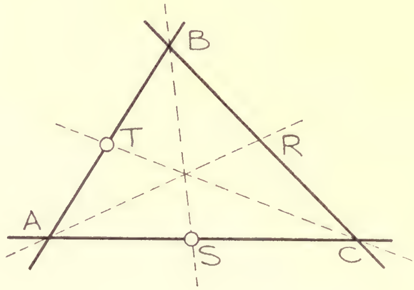
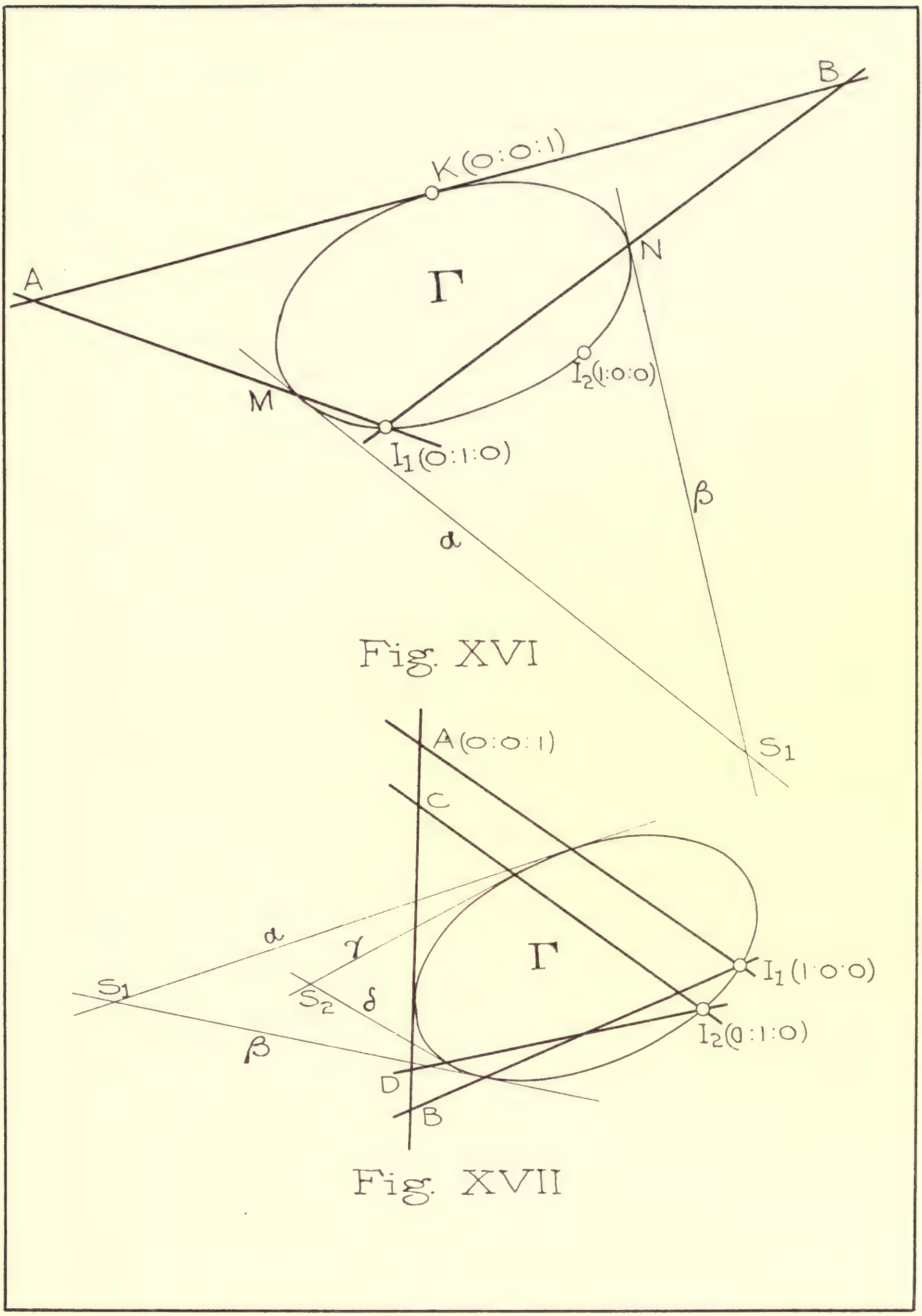


Fig. XV

Figs. 14 and 15



Figs. 16 and 17

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